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in an Infinite Conducting Slab

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Abstract

A current dipole of arbitrary orientation is imbedded in a uniform conductor bounded by infinite parallel insulating planes. The potential is the sum of the potentials due to the parallel and perpendicular components into which the dipole moment is resolved. Using the method of images, the potential due to either component is the sum of the potentials arising from two infinite uniformly spaced collinear arrays of dipoles.

Mathematically, the problem is to find an approximation to the sum of a slowly converging series of positive terms. This is done in such a way that the arithmetical work is shortened and the relative error is always less than any desired number. Curves are given which show how the errors of certain very simple approximations depend on position in the field. Equipotential surfaces are shown for the parallel component. For this case, at distances greater than twice the thickness, the equipotential surfaces are very close to the circular cylinders characteristic of a linear dipole of uniform density in an unbounded conductor.

1. The potential of a current dipole in an infinite medium

If there is a single point source of current in an infinite conducting medium, the current flows radially away from the source and its density at a distance r meters is

$$J = \frac{S}{4 \pi r^2} \qquad \text{amperes per square meter} \qquad (1)$$

where S is the strength of the source in amperes. The electric intensity due to the flow is

$$E = \frac{J}{\sigma} = \frac{S}{4 \pi \sigma r^2} \qquad (2)$$

in which σ is the conductivity. The potential at a distance r from the source is

$$V = \int_r^\infty E \, dr = \frac{S}{4 \pi \sigma} \int_r^\infty \frac{dr}{r^2} = \frac{S}{4 \pi \sigma r} \qquad (3)$$

Let the point at which the potential is to be calculated be taken as the origin; and let there be a sink of strength S at (x, y, z) and a source of equal strength at $(x + h, y, z)$. The potential at the origin is

$$V = \frac{S}{4 \pi \sigma} \left(\frac{1}{r_1} - \frac{1}{r} \right) \qquad (4)$$

where
$$r = (x^2 + y^2 + z^2)^{1/2} \; ; \; r_1 = \left\{ (x+h)^2 + y^2 + z^2 \right\}^{1/2}$$

The function r_1^{-1} may be expanded in a Taylor series as

$$\frac{1}{r_1} = \frac{1}{r} + h \frac{\partial}{\partial x} \left(\frac{1}{r} \right) + \cdots \qquad (5)$$

giving

$$V = \frac{Sh}{4 \pi \sigma r^2} \left(-\frac{x}{r} + \qquad \text{higher powers of } h \right) \qquad (6)$$

the limit of which as $h \to 0$ and $S \to \infty$, in such a way that $L(Sh) = M$, is

$$V = \frac{M \cos \theta}{4 \pi \sigma r^2} \tag{7}$$

where θ is the angle between the positive direction of the current moment, M, (sink to source), and the vector drawn from dipole to field point. The expression (7) for the potential is appropriate for the MKS units which we shall use.

2. Dipole axis parallel to the faces of an infinite conducting slab

Boundary conditions.

Let the faces of the slab be the planes $z = 0$ and $z = \frac{a}{2}$. Let the axis of the dipole be in the $+x$ direction and let the dipole, of moment M, be on the z axis at $z = b$.

We assume that there is a nonconducting medium in contact with both faces of the slab.

On account of symmetry we see that

$$V = 0 \text{ when } x = 0 \text{, (except at } y = 0 \text{, } z = b \text{)} \tag{8}$$

and because there can be no current normal to the boundaries

$$\frac{\partial V}{\partial z} = 0 \qquad \text{when} \qquad z = 0, \frac{a}{2} . \tag{9}$$

Let us regard the current density at any point as the vector sum of the dipole current density which would exist in unbounded space and the increment to current density due to the presence of boundaries. The latter is finite while the former increases without limit as we approach the dipole. It follows that, over a small spherical surface surrounding the dipole, the potential approaches that of a dipole of the same moment in unbounded space as the

radius of the sphere diminishes. Consider the space whose inner boundary is this sphere and whose outer boundaries are the two parallel planes and a circular cylinder whose axis is the z axis and whose radius is infinite. In this space, $\nabla \cdot \nabla V = 0$; and we know the value of the normal derivative of the potential over all boundaries. The potential within is uniquely determined by these conditions [1].

The same conditions are satisfied in an infinite conducting medium if there is an infinite linear array of dipoles of equal moments, M, whose axes are in the positive x direction, and whose coordinates are

$$X = 0; \quad Y = 0; \quad z = na \pm b \tag{10}$$

where n is an integer which assumes all values from minus infinity to plus infinity. The potential in this case is a periodic function of z ; but its value between $z = 0$ and $z = a/2$ is the same as that for the single dipole in the slab.

3. The potential due to an infinite uniform linear array of dipoles.

The array (10) may be regarded as two collinear arrays each of which has a uniform interval of length a between its elements. One array is displaced with respect to the other along the z axis a distance equal to 2b.

The potential due to one or the other of these arrays is the sum of terms of the form

$$V_n = \frac{M \cos \theta_n}{4\pi\sigma d_n^2} \tag{11}$$

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1. J. H. Jeans, "Electricity and Magnetism"; §§ 187, 377; Cambridge University Press, 1925.

where the potential V is referred to the nth dipole. In this expression d_n is the length of the vector from the nth dipole to the point $P(x,y,z)$ for which the potential is being calculated; and Θ_n is the angle between the axis of the dipole and this vector. In this calculation it is convenient to use a system of coordinates in which the elements of one array are at

$$X = 0 ; \quad y = 0 ; \quad ; \quad z' = na \tag{12}$$

and, as before, the positive axes of all the dipoles are in the x direction. That being the case,

$$|d_n| = \left\{ x^2 + y^2 + (z' - na)^2 \right\}^{1/2} \tag{13}$$

and the length of the projection of this vector on the axis of the nth dipole is x. Therefore,

$$\cos \Theta_n = \frac{x}{\left\{ x^2 + y^2 + (z' - na)^2 \right\}^{1/2}} \tag{14}$$

$$V_n = \frac{Mx}{4 \pi \sigma \left\{ x^2 + y^2 + (z' - na)^2 \right\}^{3/2}} \tag{15}$$

and the potential due to the whole of one array, referred to any element of the array as origin, is

$$V = \frac{Mx}{4 \pi \sigma} \sum_{n=-\infty}^{\infty} \frac{1}{\left\{ x^2 + y^2 + (z' - na)^2 \right\}^{3/2}} \tag{16}$$

Inspection of (16) shows that

$$V(x,y,z') = -V(-x,y,z') \tag{17}$$

$$V(x,y,z') = V(x,-y,z') \tag{18}$$

$$V(x,y,z') = V(x,y,-z') \tag{19}$$

because $(-z'-na)^2$ has the same value for $n = m$ as $(z'-na)^2$ has for $n = -m$.

$$V(x,y,z') = V(x,y,z'+ma); \quad m \equiv \text{any integer}; \tag{20}$$

i.e., the function is periodic in z' with period a.

$$V(x,y,z') = V(x,y,a-z') \tag{21}$$

which follows from (19) and (20), and shows that the potential is also symmetrical with respect to any plane $z' = \frac{1}{2}na$.

The relation between z' and z is either

$$z' = z-b \tag{22a}$$

or

$$z' = z + b \tag{22b}$$

depending on whether we are considering the upper or lower of the two uniform arrays. Thus if (16) is written

$$V = V(x,y,z') \tag{23}$$

we find the potential due to the dipole in the slab to be

$$V_1(x,y,z) = V(x,y,z-b) + V(x,y,z+b) \tag{24}$$

In particular,

$$V_1 = 2V(x,y,z) \qquad \text{when } b = 0 \tag{25}$$

Thus we have a formal solution of the problem. However, the right hand side of (16) is an infinite series of positive terms; and we must find out how to calculate V in such a way that the relative error is known to be within satisfactory bounds.

4. <u>Summation of the series</u> $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{\{x^2+y^2+(na-z')^2\}^{3/2}}$

We begin by transforming the series so that the summation involves only dimensionless numbers. From (16)

$$\frac{4\pi\sigma V}{Mx} = \sum_{-\infty}^{\infty} \frac{1}{\{x^2+y^2+(na-z')^2\}^{3/2}} = \frac{1}{a^3}\sum_{n=-\infty}^{\infty} \frac{1}{\left\{\frac{x^2+y^2}{a^2} + \left(\frac{n-z'}{a}\right)^2\right\}^{3/2}} \tag{26}$$

Writing

$$\frac{x}{a} \equiv p_1 \; ; \; \frac{y}{a} \equiv p_2 \quad \frac{z'}{a} \equiv q \; ; \quad p_1{}^2 + p_2{}^2 \equiv p^2 \tag{27}$$

the potential becomes

$$V = \frac{Mx}{4\pi\sigma a^3} S = \frac{Mp_1}{4\pi\sigma a^2} S \tag{28}$$

where

$$S \equiv \sum_{n=-\infty}^{\infty} \frac{1}{\{p^2 + (n-q)^2\}^{3/2}} \tag{29}$$

in which $0 < p < \infty$ and $0 \leq q \leq \frac{1}{2}$. Evidently the series diverges when both p and q vanish.

In order to investigate the question of convergence, we write S in the form

$$S = \frac{1}{(p^2 + q^2)^{3/2}} + \sum_{n=1}^{\infty} \frac{1}{\{p^2 + (n-q)^2\}^{3/2}} + \sum_{n=1}^{\infty} \frac{1}{\{p^2 + (n+q)^2\}^{3/2}} \tag{30}$$

and observe that

$$u_n \equiv \frac{1}{\{p^2 + (n-q)^2\}^{3/2}} < \frac{1}{(n-q)^3} \leq \frac{1}{(n-\frac{1}{2})^3} < \frac{1}{n^2} \; ; (n \geq 3) \tag{31}$$

We see also that

$$v_n \equiv \frac{1}{\{p^2 + (n+q)^2\}^{3/2}} < \frac{1}{(n+q)^3} \leq \frac{1}{n^3} < \frac{1}{n^2} \; ; (n > 1) \tag{32}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, it follows that the series S is uniformly convergent over the range $0 < p < \infty$, $0 \leq q \leq \frac{1}{2}$. Similarly we can show that all the series formed by termwise partial differentiation of S once or twice with respect to any of the variables p_1, p_2, q are uniformly convergent.

Consider the continuous functions

$$u(n) \equiv \frac{1}{\{p^2+(n-q)^2\}^{3/2}} \; ; \quad v(n) \equiv \frac{1}{\{p^2+(n+q)^2\}^{3/2}} \tag{33}$$

in which n is not restricted to integral values but may be any real number equal to or greater than unity. When n is an integer, $u(n)$, and $\dot{v}(n)$ assume the same values as the constants u_n, v_n. Both $u(n)$ and $v(n)$ are monotonic functions of n which approach zero as n becomes infinite. Therefore,

$$u_m > \int_m^{m+1} u(n)\, dn > u_{m+1}$$

$$u_{m+1} > \int_{m+1}^{m+2} u(n)\, dn > u_{m+2}$$

$$- - - - - - - - - - -$$

$$u_{K-1} > \int_{k-1}^{k} u(n)\, dn > u_k$$

Adding these inequalities,

$$\sum_m^{K-1} u_n > \int_m^{K} u(n)\, dn > \sum_{m+1}^{K} u_n \tag{34}$$

As $K \to \infty$, the series and the integral both converge, and

$$\sum_m^{\infty} u_n > \int_m^{\infty} u(n)\, dn > \sum_{m+1}^{\infty} u_n \tag{35}$$

Subtracting each term of the above inequalities from $\sum_m^{\infty} u_n$ we get

$$u_m > \sum_m^{\infty} u_n - \int_m^{\infty} u(n)\, dn > 0 \tag{36}$$

If $\overset{m-1}{\underset{1}{\sum}} u_n$ be added and subtracted from the middle term, we see that (36) is equivalent to

$$u_m > \sum_1^\infty u_n - \left\{ \sum_1^{m-1} u_n + \int_m^\infty u(n)\, dn \right\} > 0 \tag{37}$$

The geometrical interpretation of (36) is seen in Fig. 1 in which the areas of the rectangles (of unit width) are u_m, u_{m+1}, etc. The middle term of the inequality is evidently the sum of the areas of all the three-sided figures which lie above the continuous curve. These figures may all be slid to the left, as shown, into the rectangle whose height is u_m and for each portion of the left-hand rectangle so occupied there corresponds an unoccupied three-sided space to the left of it. Evidently, if $D_n^2 u(n) > 0$; $(m \leq n < \infty)$ the u(n) curve is concave upward; and each occupied space in the mth rectangle is greater than the corresponding unoccupied space. Accordingly if

$$D_n^2 u(n) > 0, \quad m \leq n < \infty,$$

we have

$$u_m > \sum_1^\infty u_n - \left\{ \sum_1^{m-1} u_n + \int_m^\infty u(n)\, dn \right\} > \frac{1}{2} u_m$$

or

$$\frac{1}{2} u_m > \sum_1^\infty u_n - \left\{ \sum_1^{m-1} u_n + \int_m^\infty u(n)\, dn + \frac{1}{2} u_m \right\} > 0 \tag{38}$$

when the second derivative is positive over the range of integration.

Thus (37) shows that in general the relative error in using

$$\sum_1^{m-1} u_n + \int_m^\infty u(n)\, dn$$

in place of the sum is less than the right hand term of the inequality

$$0 < \frac{\sum_1^\infty u_n - \left\{ \sum_1^{m-1} u_1 + \int_m^\infty u(n)\,dn \right\}}{\sum_1^\infty u_n} < \frac{u_m}{\sum_1^\infty u_n} < \frac{\frac{1}{2} u_m}{\sum_1^\infty u_n + \int_m^\infty u(n)\,dn} \qquad 39)$$

whereas if the above condition on the second derivative is satisfied (38) shows that the relative error in using

$$U \equiv \sum_1^{m-1} u_n + \int_r^\infty u(n)\,dr + \frac{1}{2} u_m \qquad (40)$$

instead of the sum is less than the right hand side of

$$0 < \frac{\sum_1^\infty u_n - U}{\sum_1^\infty u_n} < \frac{\frac{1}{2} u_m}{\sum_1^\infty u_n} < \frac{\frac{1}{2} u_m}{U} \qquad (41)$$

The generality and usefulness of (39) and (41) as well as the elementary character of the derivation would cause one to expect that these facts would be well known and readily available in the literature of infinite series. I have not been able to find them.

It will be seen that if the derivative terms and the remainder are omitted from the formula of Euler and Maclaurin the result is the same as (40). For the present circumstances the appropriate form of the Euler-Maclaurin formula is

$$\sum_1^\infty u_n \simeq U - \sum_{r=1}^{K-1} \frac{B_{2r}}{(2r)!} \left[D_n^{2r-1} u(n) \right]_{n=m} - R_{K-1} \qquad (42)$$

where the remainder may be written

$$R_{K-1} = \int_m^\infty \frac{\left[\bar{B}_{2K}(n) - \bar{B}_{2K}(0) \right]}{(2K)!} D_n^{2K} u(n)\,dn \qquad (43)$$

in which B and $\bar{B}$ are Bernoulli numbers and periodic Bernoulli

functions respectively.

In (42) a sufficient condition for the first term no less

$$- \frac{B_{2k}}{(2k)!} \left[D_n^{2k} u(n) \right]_{n=m}$$

to exceed the remainder, F_{k-1}, and to have the same sign as R_{k-1},

is that $D_n^{2K} u(n) \cdot D_n^{2K-2} u(n) > 0$; $(m \leq n < \infty)$

while $D_n^{2k} u(n)$ does not change sign throughout the interval

$m \leq n < \infty$ [2]. Thus, for example, the error in (40) is less than

the first derivative term

$$\left[- \frac{B_2}{2} D_n' u(n) \right]_{n=m}$$

provided $D_n^2 u(n) \cdot D_n^4 u(n) > 0$; $(n \geq m)$

and $D_n^2 u(n)$ is of uniform sign for all $n \geq m$. Similarly, if only

the first derivative term is retained in (42), the error is less

than the next term,

$$\left[- \frac{B_4}{4!} D_n^3 u(n) \right]_{n=m}$$

provided that for all $n \geq m$, $D_n^4 u(n)$ and $D_n^6 u(n)$ have the same sign

and the sign does not change.

If it is only known that $D_n^{2k} u(n)$ does not change its sign, the

error is numerically less than twice the first neglected derivative

term, and has the same sign [2].

The terms of the series (29) diminish with increasing n most

rapidly if p is small. When p is large, each dipole contributes less

potential and the contributions of the dipoles are more nearly equal.

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2. J. F. Steffensen, Interpolation, p. 133
   The Williams & Wilkins Co., Baltimore, 1927.

Furthermore, since the distance to the dipoles does change little as we vary z', while keeping $x^2 + y^2$ constant, we should expect that, with large values of p, the potential would be nearly independent of z'. These considerations tempt one to use the integral from minus infinity to infinity in place of the sum and to seek a limit to the error committed. From the physical standpoint, using the integral for the sum is equivalent to replacing the array of points with a continuous dipole line of strength $\frac{M}{a}$ per unit length along the z axis. The equipotential surfaces of such a distribution are well known to be circular cylinders which all pass through the source line and whose axes pass through the x axis parallel to the z axis.

Reasoning as we did before to establish (34), and now allowing u to represent the general term without restriction on the sign and value of n, we have

$$\int_{-\infty}^{0} u(n)\,dn - \sum_{-\infty}^{-1} u_n = \theta_1 u_o \; ; \quad (o > \theta_1 < 1)$$

$$\sum_{o}^{\infty} u_n - \int_{0}^{\infty} u(n)\,dn = \theta_2 u_o \; ; \quad (o < \theta_2 < 1)$$

Subtracting

$$\int_{-\infty}^{\infty} u(n)\,dn - \sum_{-\infty}^{\infty} u_n = u_o\,\theta \; ; \quad \theta \equiv \theta_1 - \theta_2 \; ; \quad |\theta| < 1 \tag{44}$$

The relative error in using the integral for the sum is therefore

$$\frac{\int - S}{\int - \theta u_o} = \frac{\theta u_o}{\int}\left[1 + \frac{\theta u_o}{\int} + \left(\frac{\theta u_o}{\int}\right)^2 + \cdots\right. \tag{45}$$

$$= \frac{\theta}{2p}\left(1 - \frac{3}{2}\frac{q^2}{p^2} + \frac{3\cdot5}{2\cdot4}\frac{q^4}{p^4} - \cdots\right)\left[1 + \cdots\right.$$

Since $q \leq 1/2$, and p is more confident to be large than less than unity, the relative error is approximately $\dfrac{\theta}{2p}$, which is less than $\dfrac{1}{2p}$.

In obtaining this limit, no account has been taken of any properties of the series except its monotone character. It might be supposed that one could establish directly a slight or limit for the error of the infinite integral by taking account of the shape of the curve. There are difficulties, however, which make it more convenient to determine the smaller error limit by comparing the infinite integral with the results of a calculation which uses (40) or (42). We are thus able to find the smallest value of p for which the integral will give any desired accuracy. Since we have

$$\int_{-\infty}^{\infty} u(\eta)\, d\eta \;=\; \frac{2}{\rho^{2}} \tag{46}$$

the potential is simply

$$V \;\simeq\; \frac{\left(\dfrac{M}{a}\right) x}{2\pi\sigma\,(x^{2}+y^{2})} \tag{47}$$

Even for values of p as small as unity the error in (47) at $q = 0,\ 1/2$ is only about 1 per cent. While (47) can give no information about the variation of V with q, it greatly shortens the work of plotting equipotentials and flow lines in the plane $z' = 0$.

Below certain values of p and q, which depend on the required accuracy, it is possible to use as the equations of the equipotential surfaces

$$\rho^{2} \;=\; \frac{M}{4\pi\sigma V}\,\cos\theta \;, \quad (\rho^{2} \equiv x^{2}+y^{2}+z'^{2}) \tag{48}$$

for which the streamlines are

$$\rho = A \sin^2 \theta$$

(49)

This means that we ignore all the dipoles except the nearest one. The values of p and q below which (48) is good enough are found by comparison of (48) with the results obtained by using (30) together with (40).

In the foregoing discussion the symbols $u(n)$, $v(n)$ and u_n, v_n have been used to distinguish between the two forms which the general series term assumes when n is kept positive. Evidently all that has been said about $u(n)$ and u_n from (34) to (43) inclusive applies as well to $v(n)$ and v_n.

5. Collected explicit formulas for calculating the potential

(A)
$$V \simeq \frac{\left(\frac{M}{a}\right) x}{2 \pi \sigma \left(x^2 + y^2\right)}$$

(47)

This formula does not contain z'. It is most accurate when the distance from the dipole is large (Fig. 2).

(B)
$$V \simeq \frac{M x}{4 \pi \sigma \left(x^2 + y^2 + z'^2\right)^{3/2}}$$

(48)

Formula (B) is most accurate when the distance from the dipole is small compared with the thickness of the slab (Fig. 3).

(C)
$$V = \frac{M x}{4 \pi \sigma a^3} \left[\sum_{n=0}^{m-1} \frac{1}{\{p^2 + (n-q)^2\}^{1/2}} + \frac{1}{p^2}\left[1 - \frac{m-q}{\{p^2 + (m-q)^2\}^{1/2}}\right] \right.$$
$$\left. + \frac{1}{2\{p^2 + (m-q)^2\}^{3/2}} + \sum_{n=1}^{n-1} \frac{1}{\{p^2 + (n+q)^2\}^{1/2}} + \frac{1}{p^2}\left[1 - \frac{m+q}{\{p^2 + (m+q)^2\}^{1/2}}\right] \right.$$
$$\left. + \frac{1}{2\{p^2 + (m+q)^2\}^{3/2}} + R_o \right]$$

(30),
(40)

Various limits for the remainder in (C).

(i) $\quad 0 < R. < \frac{1}{2}\left[\frac{1}{\{p^2+(m-q)^2\}^{3/2}} + \frac{1}{\{p^2+(m+q)^2\}^{3/2}}\right]; \quad (m-q \geq \frac{p}{2})$

(ii) $\quad 0 < R. < 2 \cdot \frac{1}{4}\left[\frac{m-q}{\{p^2+(m-q)^2\}^{3/2}} + \frac{m+q}{\{p^2+(m+q)^2\}^{3/2}}\right]; \quad (m-q \geq \frac{p}{2})$

(iii) $\quad 0 < R. < \frac{1}{4}\left[\frac{m-q}{\{p^2+(m-q)^2\}^{3/2}} + \frac{m+q}{\{p^2+(m+q)^2\}^{3/2}}\right]; \quad (m-q \geq 1.2p)$

This formula and the Euler-Maclaurin formula (D) which follows are always applicable.

(D) $\quad V = \frac{Mx}{4\pi\sigma a^3}\left[\left[\sum_{n=0}^{m-1}\frac{1}{\{p^2+(n-q)^2\}^{3/2}} + \frac{1}{p^2}\left[1 - \frac{m-q}{\{p^2+(m-q)^2\}^{1/2}}\right]\right.\right.$

$\quad + \frac{1}{2\{p^2+(m-q)^2\}^{3/2}} + \sum_{n=1}^{m-1}\frac{1}{\{p^2+(n+q)^2\}^{3/2}} + \frac{1}{p^2}\left[1 - \frac{m+q}{\{p^2+(m+q)^2\}^{1/2}}\right]$

$\quad + \frac{1}{2\{p^2+(m+q)^2\}^{3/2}} + \frac{1}{4}\left[\frac{m-q}{\{p^2+(m-q)^2\}^{5/2}} + \frac{m+q}{\{p^2+(m+q)^2\}^{5/2}}\right] + R_1\Big]\Big]$

$$(42)$$

The remainder in (D) satisfies the inequality

$-2 \cdot \frac{1}{48}\left[(m-q)\frac{4(m-q)^2 - 3p^2}{\{p^2+(m-q)^2\}^{9/2}} + (m+q)\frac{4(m+q)^2 - 3p^2}{\{p^2+(m+q)^2\}^{7/2}}\right] < R_1 < 0$

provided (m-q) $> 1.202p$.

6. The Electric Field

The series which result from termwise differentiation of the

S series with respect to p_1, p_2, and q are uniformly convergent. The order of differentiation and summation may, therefore, be inverted, and we have

$$\frac{\partial S}{\partial p_1} = \frac{\partial}{\partial p_1} \sum_{n=-\infty}^{\infty} u_n = \sum_{n=-\infty}^{\infty} \frac{\partial u_n}{\partial p_1} = -3 p_1 \sum_{n=-\infty}^{\infty} t_n ; \left[t_n \equiv \{ p^2 + (n-q)^2 \}^{-\frac{5}{2}} \right] \quad (50)$$

$$\frac{\partial S}{\partial p_2} = -3 p_2 \sum_{n=-\infty}^{\infty} t_n \quad (51)$$

$$\frac{\partial S}{\partial q} = 3 \sum_{n=-\infty}^{\infty} (n-q) t_n \quad (52)$$

Using (28), the electric intensities are therefore

$$E_x = -\frac{\partial V}{\partial x} = -\frac{\partial V}{\partial p_1} \frac{\partial p_1}{\partial x} = -\frac{M}{4\pi\sigma a^3} \left(p_1 \frac{\partial S}{\partial p_1} + S \right) = \frac{M}{4\pi\sigma a^3} \left(3 p_1^2 \sum_{-\infty}^{\infty} t_n - S \right) \quad (53)$$

$$E_y = \frac{M}{4\pi\sigma a^3} \cdot 3 p_1 p_2 \sum_{-\infty}^{\infty} t_n \quad (54)$$

$$E_z = -\frac{M}{4\pi\sigma a^3} \cdot 3 p_1 \sum_{-\infty}^{\infty} (n-q) t_n \quad (55)$$

In the region in which the approximation formula (47) is accurate enough, the electric intensities become

$$E \simeq \frac{\left(\frac{M}{a}\right)}{2\pi\sigma r^2} \; ; \; E_\theta \simeq \frac{\left(\frac{M}{a}\right) \sin\phi}{2\pi\sigma r^2} \; ; \; E_r \simeq \frac{\left(\frac{M}{a}\right) \cos\phi}{2\pi\sigma r^2} \quad (56)$$

$$E_x \simeq \frac{\left(\frac{M}{a}\right)}{2\pi\sigma r} \left(\cos^2\phi - \sin^2\phi \right) = \frac{M}{2\pi\sigma a^3} \cdot \frac{p_1^2 - p_2^2}{p^4} \quad (57)$$

$$E_y \simeq \frac{\left(\frac{M}{2}\right)}{2\pi\sigma r^2} \sin 2\phi = \frac{M}{\pi\sigma a^3} \cdot \frac{b\,\rho_z}{\rho^4} \quad ; \quad E_z \simeq 0 \tag{58}$$

Formulas (56)-(58) are applicable when p is large enough. In these three equations ϕ is $\tan^{-1} y/x$ and r^2 is $x^2 + y^2$.

Near enough to the origin one may use the approximation given in (48) and obtain

$$E_\rho \simeq \frac{M\cos\theta}{2\pi\sigma\rho^3} \quad ; \quad \left(\rho^2 \equiv x^2 + y^2 + z'^2\right) \tag{59}$$

$$E_\theta \simeq \frac{M\sin\theta}{4\pi\sigma\rho^3} \tag{60}$$

In (59) and (60) the angle θ is not the conventional polar distance of spherical polar coordinates. It is measured between the x direction and the direction of the field point.

Between the near and far regions, the above approximations are not good enough and the fields must be computed in some other way such as is shown in (53), (54) and 55). The series $\sum\limits_{-\infty}^{\infty} t_n$ and $\sum\limits_{-\infty}^{\infty} (n-q)t_n$ converge more rapidly than the series for the potential; and methods like those used for the potential may be used for calculating the sums with any desired accuracy.

7. Dipole axis normal to the faces

Let the dipole moment, M, be in the z direction. Using the method of images, we find the potential within the slab to be the same as that due to two oppositely directed infinite collinear arrays whose elements are at

$$x = 0, \ y = 0, \ z' = na; \ (z' \equiv z-b)$$

and

$$x = 0, \ y = 0, \ z'' = na; \ (z'' \equiv z+b)$$

The potential due to the first array, whose moments are in the positive z direction, is

$$V_{(+)} = \frac{N}{4\pi\sigma} \sum_{n=-\infty}^{\infty} \frac{z'-na}{\{x^2+y^2+(z'-na)^2\}^{3/2}}$$

$$= -\frac{N}{4\pi\sigma a^2} \sum_{-\infty}^{\infty} \frac{(n-q')}{\{p^2+(n-q')^2\}^{3/2}} \; ; \; \left(q' \equiv \frac{z'}{a} \right) \tag{61}$$

and that due to the other array, the moments of whose elements are in the negative z direction, is

$$V_{(-)} = \frac{N}{4\pi\sigma a^2} \sum_{-\infty}^{\infty} \frac{(n-q'')}{\{p^2+(n-q'')^2\}^{3/2}} \; ; \; \left(q'' \equiv \frac{z''}{a} \right) \tag{62}$$

Hence the total potential of the dipole is

$$V_1 = V_{(-)}(p, q'') - V_{(-)}(p, q') \tag{63}$$

Evidently V_1 vanishes everywhere if $b = 0$, $a/2$. We may write the sum in (61), (62) as follows:

$$\sum_{-\infty}^{\infty} \frac{n-q}{\{p^2+(n-q)^2\}^{3/2}} = \sum_{-\infty}^{\infty} \frac{n}{\{p^2+(n-q)^2\}^{3/2}} - q \, S$$

$$= \sum_{n=1}^{\infty} \frac{n}{\{p^2+(n-q)^2\}^{3/2}} - \sum_{n=1}^{\infty} \frac{n}{\{p^2+(n+q)^2\}^{3/2}} - q \, S \tag{64}$$

where S is the sum which appears in the potential of the parallel component.

8. Numerical data

Fig. 2 shows how the relative error of formula A depends on p and q. The ordinate is, of course, not the actual relative error

because this number is unattainable. The number plotted against p, with q as parameter, is $100(V_C - V_A)/V_C$ in which an upper limit of the error of V_C, as found from (C,111), is kept less than 10% of $\left| V_C - V_A \right|$. Thus the sign of the plotted ordinate is always the same as that of the actual relative error. The magnitude is greater than that of the relative error when $V_C - V_A \angle 0$. When $V_C - V_A > 0$, it is possible that the ordinates of the curves may be a little less than the relative error. The subscripts of V used here refer to the working formulae (A) to (D).

In Fig. 3 we show a similar measure of the relative error in using formula B which is plotted against p for various values of q. The ordinate is $100(V_C - V_B)/V_C$. In this case the error limit of V_C, calculated by means of (C,1) is always less than 1% of $\left| V_C - V_B \right|$.

It is unnecessary to exhibit equipotential curves of the parallel component for p greater than about 1.0 or 1.5 because Fig. 2 shows that the equipotential surfaces, $p_1 S = k$ (a constant), are very accurately represented by circular cylinders whose axes, parallel to the z axis, pass through the point, $(1/k, 0)$, and whose radius is $1/k$.

The nature of the equipotential surfaces of the parallel component in the region $0 \leq p \leq 1$, in part of which computation by formula C is necessary, is shown by means of their curves of intersection with the planes q = 0, 1/4, 1/2 in Figs. 4, 5, 6. Near the origin the data of Fig. 3 enable us to use the simple formula B. The parameters attached to the individual curves are the values of the dimensionless number $p_1 S$ (see equation 28) from

which the potential may be obtained by multiplying by 6×10^2. The data of Figs. 4, 5, and 6 are not in error by more than two per cent. No equipotential surfaces have been plotted for the perpendicular component.

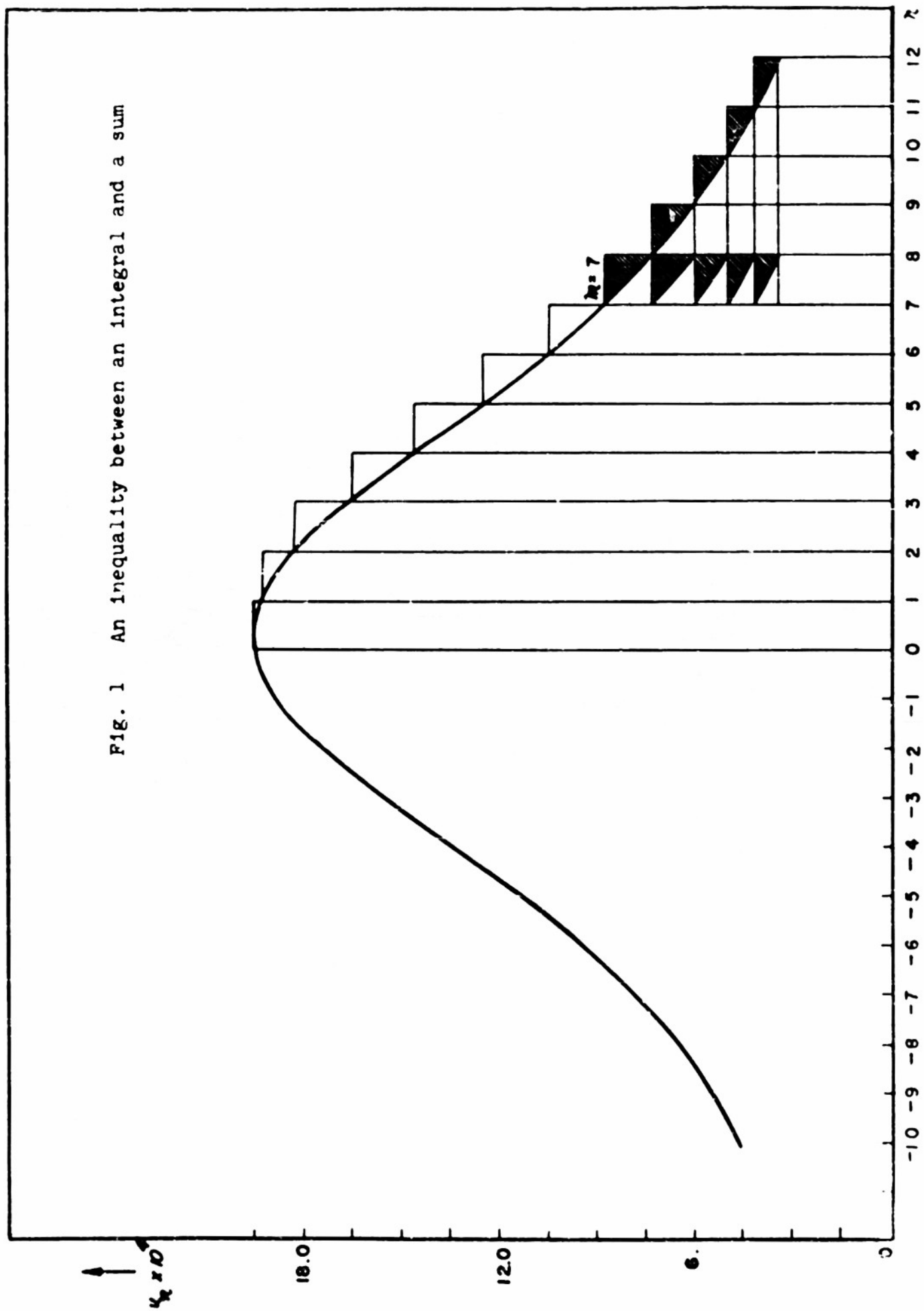I am indebted to Mrs. Lois Edelstein for doing all the numerical work and for verifying part of the analysis.

Fig. 1   An inequality between an integral and a sum

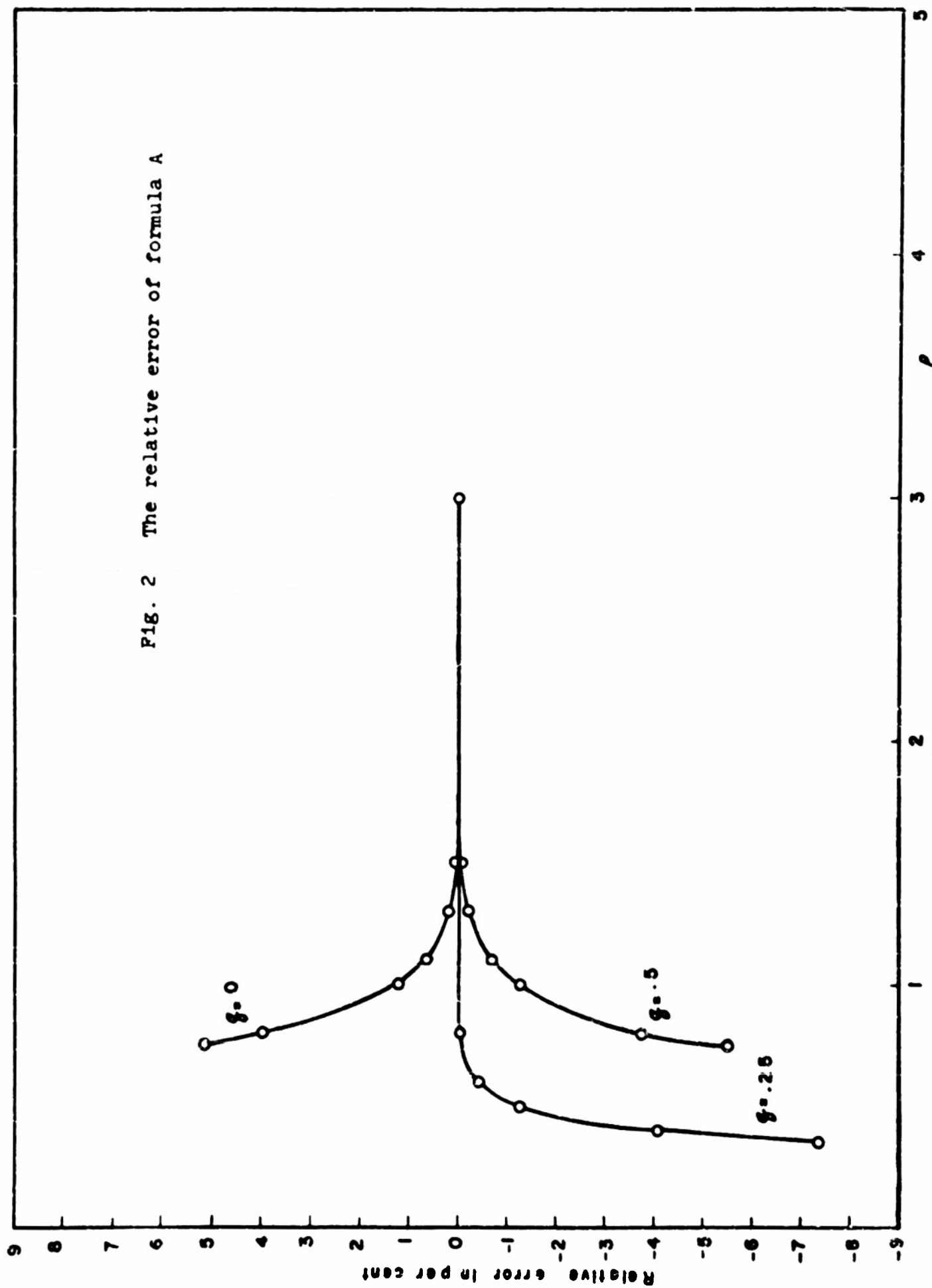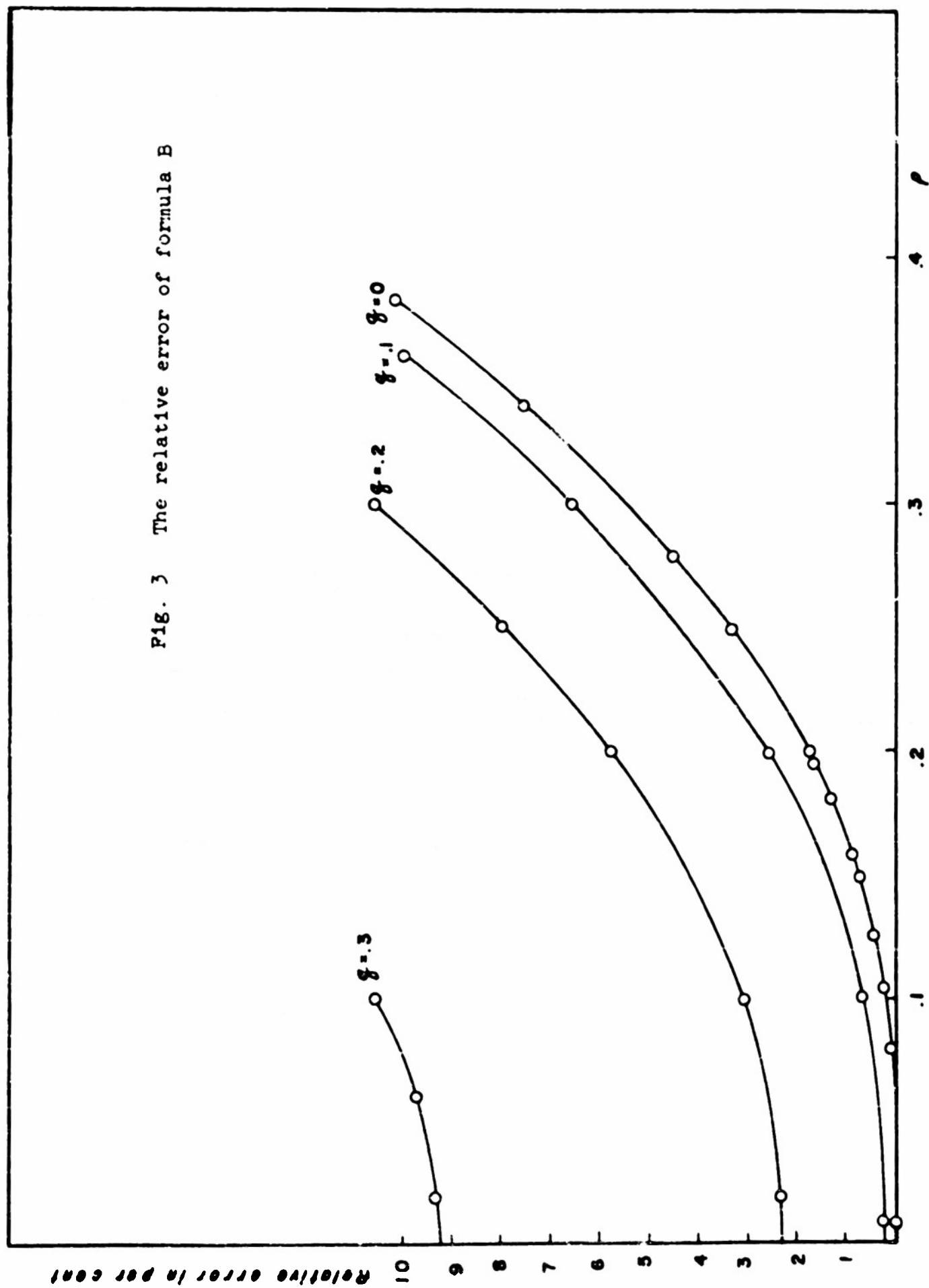Fig. 2   The relative error of formula A

Fig. 3   The relative error of formula B

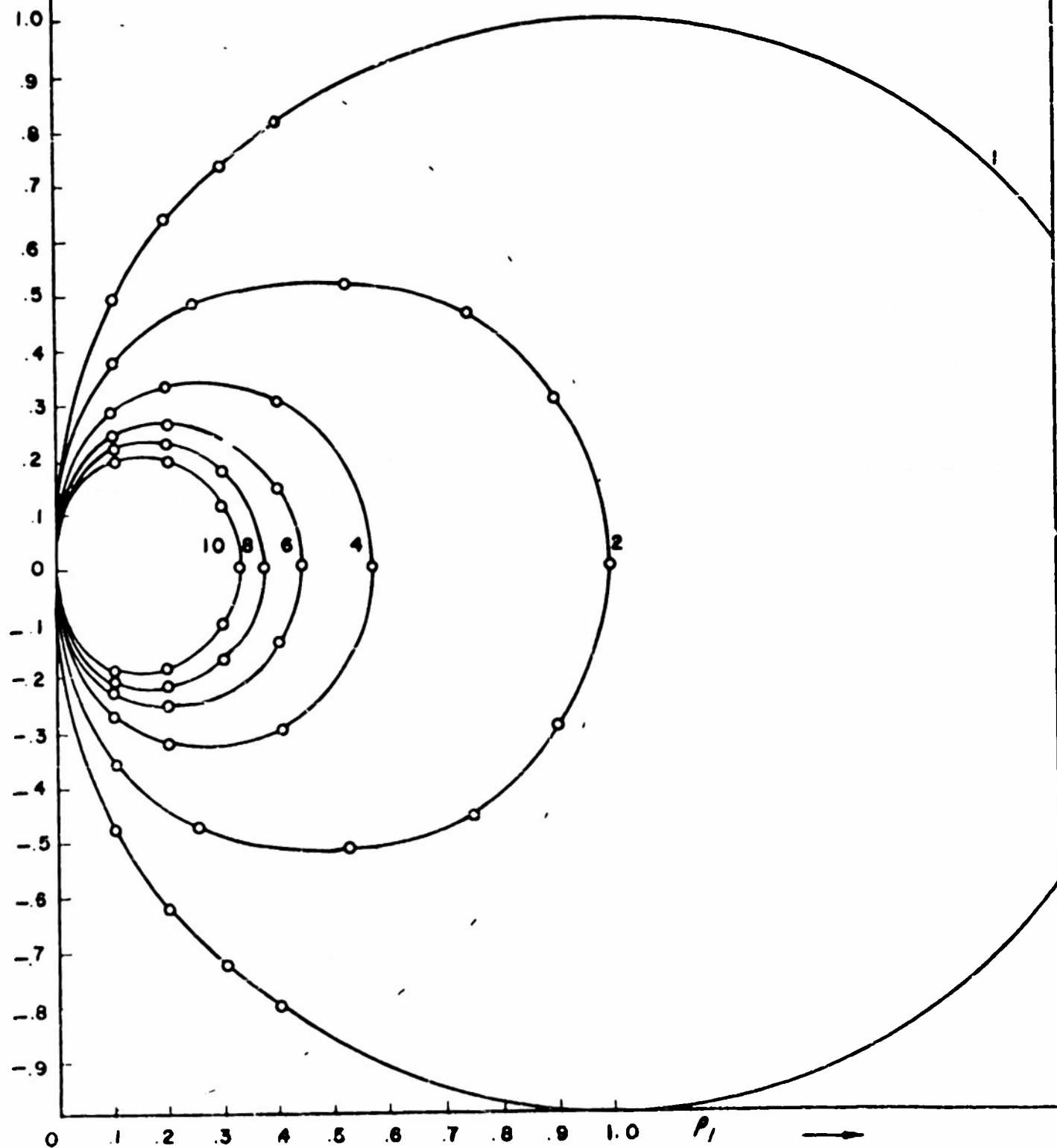Fig. 4   Intersections of equipotential surfaces with the plane q = 0. Horizontal dipole.

Fig. 5   Intersections of equipotential surfaces with the
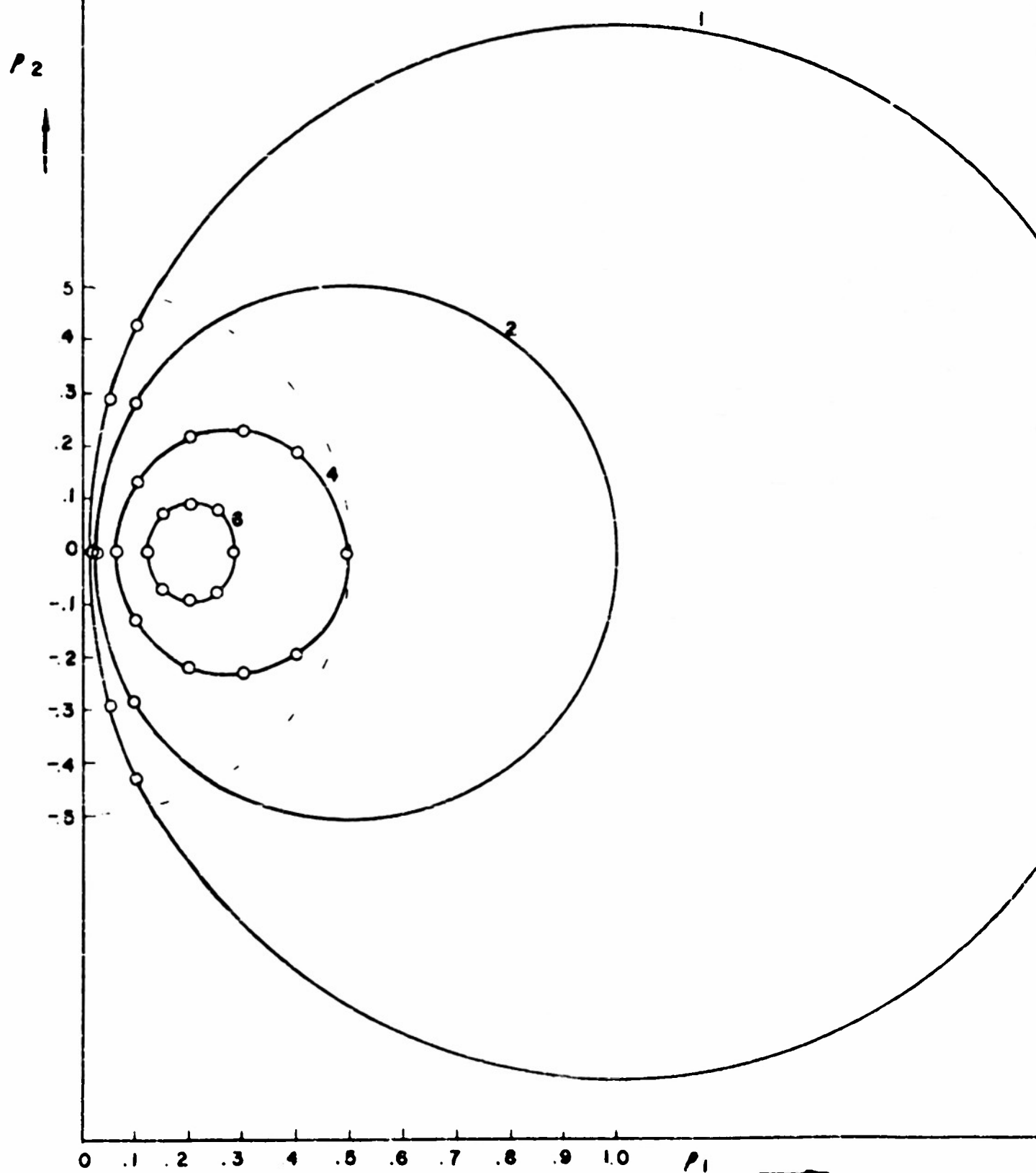plane q = 1/4.   Horizontal dipole.

Fig. 6   Intersections of equipotential surfaces with the plane q = 1/2.   Horizontal dipole.